Linearized pressure distributions with strong supersonic entropy layers

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The pressure distribution which results from the disturbance of initially plane, parallel, and constant-pressure inviscid supersonic flow in which there is a strong entropy variation near a surface is investigated. The disturbance considered may arise from a small deflection of the surface or may be the result of a simple wave impinging on the entropy layer. This linear problem has been treated previously by a somewhat different technique for transonic entropy layers by Howarth (1948), Tsien & Finston (1949), and Lighthill (1950, 1953). The results obtained in this investigation are useful principally in that they permit the general behaviour of such pressure distributions to be deduced without resort to the more general method of characteristics for rotational flows.

1. Introduction

One of the principal features of hypersonic wing theory is the fact that, in general, shock-expansion theory in the sense that it is used at low supersonic speeds is inapplicable. One reason for this is that expansion waves arising at the surface are reflected by the shock wave soon enough to affect the surface pressure. This phenomenon has been adequately treated in the literature (see, for example, Scheuing *et al.* 1959). Another reason for the failure of the usual shock-expansion theory is the existence of entropy gradients near the surface of wings with blunt noses. These gradients arise from the fact that the air near the surface has passed through a normal or nearly normal shock associated with the blunt nose, while the air farther from the surface has passed through a weaker oblique shock. These gradients alter the strength of expansion waves arising at the surface and produce reflections that alter the surface pressure. The case of weak gradients has also been treated in, for example, Scheuing *et al.* (1959). This paper is concerned with the effect of strong gradients.

If one considers the general behaviour of the propagation into an otherwise undisturbed flow of an expansion wave caused by a small surface deflection when there exists above the surface a strong variation of entropy, i.e. a strong variation of local Mach number from the Mach number at the surface M_0 to some higher Mach number at the edge of the layer M_δ, one finds that the drop in surface pressure consists of two parts:

(1) The usual instantaneous Prandtl–Meyer drop in pressure due to the surface deflection. The magnitude of this pressure drop is that appropriate to the local surface Mach number M_0.

(2) A gradual drop in pressure after the initial drop until is it complete at a value of pressure appropriate to a Prandtl–Meyer expansion around the surface corner at the external Mach number M_δ. This approach to the proper final pressure is exponential in character.

The present paper will be concerned with a study of the character of such exponential 'tails'. The present work arose quite naturally out of a more general study of pressure distributions about hypersonic bodies.

In what follows, we shall discuss in some detail the nature of the downstream influence of small pressure disturbances which propagate into an otherwise undisturbed, inviscid, entirely supersonic flow near a surface above which exists a strong variation of Mach number within a finite region. This problem is very closely related to previous work carried out by Howarth (1948), Tsien & Finston (1949), and Lighthill (1950, 1953). In these papers the authors were primarily concerned with the behaviour of shock boundary-layer interactions and in particular with the upstream influence of a pressure disturbance introduced into an otherwise undisturbed flow having a variation of Mach number normal to the stream lines. In particular, Howarth considered the problem of a supersonic wavelet arising in the region $0 < y < \infty$, where the Mach number was constant and larger than unity, impinging on the semi-infinite region $-\infty < y < 0$, where the Mach number was constant and less than unity. Tsien & Finston carried on in this vein and studied the wave impingement problem in the upper half-plane $0 < y < \infty$ such that the surface $y = 0$ was a streamline and in the region $0 < y < \delta$ the Mach number was constant and less than unity, while for the rest of the half-plane $\delta < y < \infty$ the Mach number was constant and larger than unity. Tsien & Finston considered for such a distribution of Mach number the case of an impinging wavelet as well as the pressure disturbance arising from a small deflection of the surface streamline at $y = 0$. Lighthill (1950) further generalized the problem by considering that the distribution of Mach number in the region $0 < y < \delta$ varied continuously and monotonically from zero at $y = 0$ to a constant supersonic Mach number at $y = \delta$. Later Lighthill (1953), in order to make his findings more compatible with experimental results, considered the case just described but with $M \neq 0$ at $y = 0$. In these studies the flow was considered to be inviscid except in the case of Lighthill (1953), where the effects of viscosity in a small sublayer near the surface are discussed.

As will be seen in what follows, these papers are all very closely related to the problem considered here. The method of treatment is, however, somewhat different. This is because earlier authors were interested in mixed subsonic and supersonic flows and hence in both the upstream and downstream influence of pressure disturbances. In the present paper, in which the flow is considered to be entirely supersonic, only the downstream influence need be considered. Hence, one-sided Laplace transforms are adequate in this paper where the earlier authors used two-sided Fourier transforms. The theoretical treatment in our paper differs from that of Lighthill (1950, 1953) primarily in detail and emphasis.

Instead of being interested in upstream influence in a subsonic layer we are interested in downstream influence in a completely supersonic flow. Instead of being interested in approximate values of a series of real poles of an analytic function we are interested in the precise values and residues at one real pole and a number of complex ones. The eventual aim in our case is to provide a significant but simple improvement of the shock-expansion method of calculating pressures on a body.

2. Derivation of basic equations

The two-dimensional equations of motion (see Hayes & Probstein 1959) in streamline co-ordinates, ξ and n, respectively along and normal to the streamlines, are

$$p_\xi + \rho q q_\xi = 0, \tag{2.1}$$

$$p_n + \rho q^2 \theta_\xi = 0, \tag{2.2}$$

and the equation of continuity is

$$(\rho q)_\xi + \rho q \theta_n = 0, \tag{2.3}$$

where p is the pressure, ρ the density, q the magnitude of the velocity, and θ its angle; subscripts denote partial differentiation. Because only infinitesimal changes from parallel flow along a flat plate are to be considered, the (ξ, n) co-ordinate system is an ordinary Cartesian system, and n can be defined as distance from the plate.

Since entropy is constant along streamlines

$$p_\xi = a^2 \rho_\xi, \tag{2.4}$$

where a is the speed of sound. Elimination of ρ_ξ and q_ξ from equations (2.1), (2.3), and (2.4) gives

$$p_\xi + \frac{\rho q^2}{M^2 - 1} \theta_n = 0, \tag{2.5}$$

where $M = q/a$. Introducing the effective ratio of specific heats defined by

$$\gamma_e = \rho a^2 / p \tag{2.6}$$

and the interaction parameter

$$\Gamma = \gamma_e M^2 / (M^2 - 1)^{\frac{1}{2}}, \tag{2.7}$$

equations (2.5) and (2.2) become

$$P_\xi + \Gamma \theta_n / (M^2 - 1)^{\frac{1}{2}} = 0, \tag{2.8}$$

$$P_n + (M^2 - 1)^{\frac{1}{2}} \Gamma \theta_\xi = 0, \tag{2.9}$$

where $P = \ln (p/p_i)$, p_i being a reference pressure.

It is assumed that M and γ_e (and hence Γ) are independent of ξ, and a new variable y is introduced by

$$dy = (M^2 - 1)^{\frac{1}{2}} dn.$$

(It is assumed that the Mach number, M, is everywhere greater than 1. In the (ξ, y)-plane the characteristics are everywhere at angles of $\pm 45°$ to the co-ordinate axes.) Equations (2.8) and (2.9) become

$$P_\xi + \Gamma \theta_y = 0, \tag{2.10}$$

$$P_y + \Gamma \theta_\xi = 0, \tag{2.11}$$

where $\Gamma = \Gamma(y)$ is known from the upstream conditions, and hence the equations are linear.

We now assume a finite entropy layer, i.e. that there is a value of y beyond which Γ is constant. Thus $\Gamma(y) = \Gamma_\infty$ for $y > h$. We are now interested in the solution of equations (2.10) and (2.11) in the semi-infinite strip $0 < y < h$ and $\xi > 0$.

In order to get the boundary conditions to be applied to this strip, consider the solution of equations (2.10) and (2.11) in the region $y > h$ where $\Gamma \equiv \Gamma_\infty$. The general solution is

$$\theta = f(\xi - y) + g(\xi + y), \tag{2.12}$$

$$P = \Gamma_\infty[f(\xi - y) - g(\xi + y)]. \tag{2.13}$$

The solution as expected consists of a set of waves moving away from the surface $y = 0$ described by $f(\xi - y)$ and a set of incoming waves described by $g(\xi + y)$.

Two separate basic problems may now be distinguished. The first problem is that of the pressure distribution resulting from a perturbation of the surface at $y = 0$ in the absence of incoming waves. In this case we get $g(\xi + y) = 0$ and $\theta(\xi, 0) = F(\xi)$ for $\xi > 0$. The second basic problem is that of finding the pressure distribution due to a given incoming disturbance $g(\xi + y)$ for $y > h$ with no surface distortion, i.e. $\theta(\xi, 0) \equiv 0$.

In what follows, we will consider these two problems in detail. The general formulation of the combined problem is given below. We assume the incoming disturbance is such that $g(\xi + h) = 0$ for $\xi \leqslant 0$ and is such that $g(\xi + h) = G(\xi)$ for $\xi > 0$. The boundary conditions to be applied to equations (2.10) and (2.11) are then

$$\left.\begin{aligned} \theta(0, y) &= 0 \qquad (0 < y < h), \\ P(0, y) &= 0 \qquad (0 < y < h), \\ \theta(\xi, 0) &= F(\xi) \quad (0 < \xi), \\ P(\xi, h) &= \Gamma_\infty[\theta(\xi, h) - 2G(\xi)]. \end{aligned}\right\} \tag{2.14}$$

This last condition arises from the fact that P and θ must be continuous at $y = h$ and is obtained from equations (2.12) and (2.13) by eliminating $f(\xi - y)$ at $y = h$.

If a function $Y(\xi, y)$ is now introduced such that

$$P = \Gamma Y_y, \quad \theta = -Y_\xi, \tag{2.15}$$

then equation (2.10) is satisfied automatically and equation (2.11) becomes

$$(\Gamma Y_y)_y - \Gamma Y_{\xi\xi} = 0, \tag{2.16}$$

and the boundary conditions (2.14) become

$$\left.\begin{aligned} Y(0, y) &= 0 \qquad\qquad (0 < y < h), \\ Y_\xi(0, y) &= 0 \qquad\qquad (0 < y < h), \\ Y(\xi, 0) &= -\int_0^\xi F(\xi)\, d\xi \quad (\xi > 0), \\ Y_y(\xi, h) + Y_\xi(\xi, h) &= -2G(\xi) \qquad (\xi > 0), \end{aligned}\right\} \tag{2.17}$$

if Γ is continuous at h. If there is a discontinuity in Γ at h, the last of these equations must be replaced by

$$\Gamma(h-)\,Y_y(\xi,h)+\Gamma_\infty Y_\xi(\xi,h) = -2\Gamma_\infty G(\xi) \quad (\xi > 0). \tag{2.17a}$$

In what follows this alternate form of the boundary condition at $y = h$ is not explicitly mentioned although it is used in some of the examples.

Let $\overline{Y}$ be the Laplace transform of Y with respect to ξ,

$$\overline{Y}(s,y) = \int_0^\infty e^{-s\xi}\,Y(\xi,y)\,d\xi.$$

Equations (2.16) and (2.17) transform to

$$(\Gamma\overline{Y}_y)_y - s^2\Gamma\overline{Y} = 0, \tag{2.18}$$

$$\overline{Y}(s,0) = -\overline{F}(s)/s, \tag{2.19}$$

$$\overline{Y}_y(s,h)+s\,\overline{Y}(s,h) = -2\overline{G}(s), \tag{2.20}$$

where $\overline{F}(s)$ and $\overline{G}(s)$ are the Laplace transforms of $F(\xi)$ and $G(\xi)$ respectively.

In brief, we seek, for a given distribution $\Gamma(y)$, the inverse transform of the solution of equation (2.18) subject to conditions (2.19) and (2.20). In particular, we are interested in the behaviour of the pressure at the wall, which is given by

$$P(\xi,0) = \Gamma(0)\,Y_y(\xi,0).$$

For the most part, because of the physical considerations discussed earlier, we consider functions $\Gamma(y)$ which are non-decreasing for y increasing, and which have strong variation, i.e. $\Gamma_\infty/\Gamma(0)$ not close to one.

3. Pressure distribution due to the small deflexion of a flat plate

3.1. *Formulation*

We now proceed to a discussion of the first basic problem mentioned in the previous section, namely that of finding the pressure distribution downstream of a small surface deflection ϵ in an otherwise flat plate. If we set $G(\xi) = 0$, and $F(\xi) = \epsilon 1(\xi)$, where $1(\xi)$ is the unit step function, the governing equations are

$$(\Gamma\overline{Y}_y)_y - s^2\Gamma\overline{Y} = 0, \tag{3.1}$$

$$\overline{Y}(s,0) = -\epsilon/s^2, \tag{3.2}$$

$$\overline{Y}_y(s,h)+s\,\overline{Y}(s,h) = 0. \tag{3.3}$$

This boundary-value problem in $Y(\xi,y)$ can be interpreted as that of a semi-infinite vibrating string where Y is the displacement at any time, ξ, and position, y, and both the density and tension of the string vary along its length according to the same function $\Gamma = \Gamma(y)$. The boundary conditions state that initially the string is at rest with zero displacement; at time $\xi = 0$ the end $y = 0$ is made to move with velocity $-\epsilon$; and the part $y > h$ has no incoming waves.

3.2. *Explicit solutions*

If the function Γ is such that

$$\frac{1}{2}\frac{\Gamma''}{\Gamma}-\frac{1}{4}\left(\frac{\Gamma'}{\Gamma}\right)^2 = c \quad (0 < y < h), \tag{3.4}$$

where c is a constant, the general solution of equation (3.1) is

$$\overline{Y}(s,y) = \Gamma^{-\frac{1}{2}}[A(s)\exp\{(s^2+c)^{\frac{1}{2}}y\}+B(s)\exp\{-(s^2+c)^{\frac{1}{2}}y\}], \tag{3.5}$$

where A and B are readily evaluated from the boundary conditions. The inverse transforms of these functions usually involve convolutions of Bessel functions and the condition (3.4) is not sufficiently general to warrant the presentation of the results. However, there are two cases that give $c = 0$, so that the inverse transforms are quite simple; these shed a good deal of light on the general case.

The first of these is given by a simple step function for Γ

$$\left.\begin{aligned}\Gamma &= \Gamma_b \quad (0 \leqslant y < h),\\ \Gamma &= \Gamma_\infty \quad (h < y).\end{aligned}\right\} \tag{3.6}$$

Then
$$\overline{Y}(s,y) = \frac{\epsilon}{s^2}\left(\frac{e^{sy}-e^{-sy}}{1-G\,e^{-2hs}} - e^{sy}\right) \quad (0 \leqslant y \leqslant h), \tag{3.7}$$

FIGURE 1. Surface pressure distribution downstream from a corner for a step Γ-distribution (equation (3.6)) with $\Gamma_\infty/\Gamma_b = 4$.

where $G = (\Gamma_\infty - \Gamma_b)/(\Gamma_\infty + \Gamma_b)$, and it is found that

$$P(\xi, 0) = \epsilon\Gamma_b[1(\xi)+2G1(\xi-2h)+2G1(\xi-4h)+\ldots], \tag{3.8}$$

a result easily found by more direct methods. This can also be written since $|G| < 1$,

$$P(\xi,0) = \epsilon\Gamma_b\left(2\frac{1-G^{n+1}}{1-G}-1\right) = \epsilon[\Gamma_\infty - (\Gamma_\infty - \Gamma_b)\,G^n] \tag{3.9}$$

for $2nh < \xi < 2(n+1)h$. This function is illustrated in figure 1 for $\Gamma_\infty/\Gamma_b = 4$.

The second case is for Γ given by

$$\left.\begin{aligned}\Gamma &= \Gamma_b(1+y/a)^2 \quad (0 \leqslant y \leqslant h),\\ \Gamma &= \Gamma_\infty \quad\quad\quad\quad (h \leqslant y),\end{aligned}\right\} \tag{3.10}$$

where $a > 0$ is taken such that

$$\Gamma_b(1+h/a)^2 = \Gamma_\infty,$$

so that Γ is continuous. Then

$$\overline{Y} = \frac{\epsilon a}{s^2}\,\frac{e^{s(y-2h)}-[1-2(a+h)\,s]\,e^{-sy}}{(a+y)\,[1-2(a+h)\,s-e^{-2hs}]} \tag{3.11}$$

and the wall pressure can be expressed by

$$P(\xi, 0) = \epsilon\Gamma_b\left[\frac{\xi}{a}+1-2\sum_{k=1}^{n}(-1)^{k-1}L_k\right] \tag{3.12}$$

for $2nh \leqslant \xi < 2(n+1)h$, where

$$L_k = \int_0^{X_k}\frac{\lambda^{k-1}e^{\lambda}}{(k-1)!}d\lambda$$

and
$$X_k = \frac{\xi-2kh}{2(a+h)}.$$

This function is illustrated in figure 2 for $\Gamma_\infty/\Gamma_b = 4$.

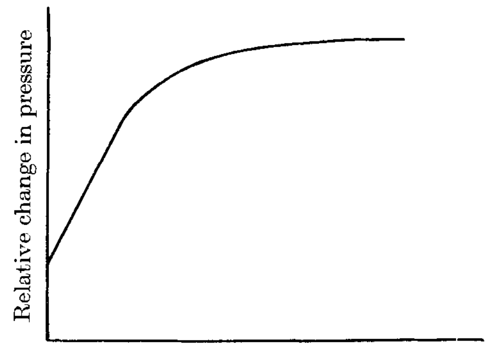

FIGURE 2. Surface pressure distribution downstream from a corner for the Γ-distribution given by equation (3.10) with $\Gamma_\infty/\Gamma_b = 4$.

3.3. *Eigenfunction expansion of solution*

A powerful method of obtaining the inverse Laplace transform of a function $f(s)$ is to sum the residues of the function $e^{s\xi}f$ at the poles of f. In order to determine the poles of $\overline{Y}$ consider the function $\Psi(s,y)$ defined by

$$(\Gamma\Psi_y)_y - s^2\Gamma\Psi = 0, \tag{3.13}$$

$$\Psi_y(s,h)+s\Psi(s,h) = 0, \tag{3.14}$$

$$\Psi(s,h) = 1. \tag{3.15}$$

Since this differential equation and one of the boundary conditions apply also to $\overline{Y}$ (equations (3.1) and (3.3)), it is easily shown that the ratio $\overline{Y}/\Psi$ is independent of y. Thus

$$\frac{\overline{Y}(s,y)}{\Psi(s,y)} = \frac{\overline{Y}(s,0)}{\Psi(s,0)},$$

or using equation (3.2),

$$\overline{Y}(s,y) = -\frac{\epsilon}{s^2}\frac{\Psi(s,y)}{\Psi(s,0)}. \tag{3.16}$$

It is seen that the zeros, in the complex s-plane, of $\Psi(s,0)$ correspond to poles of $\overline{Y}(s,y)$. Hence, we are led to consider the eigenvalue problem defined by equations (3.13) and (3.14) and

$$\Psi(s,0) = 0. \tag{3.17}$$

This differs from the usual eigenvalue problem in that the eigenvalue s occurs explicitly in the boundary condition (3.14) as well as in the differential equation.

The pressure distribution downstream of the surface deflection is now given by the sum of the residues of the function

$$\Gamma(0)\,\bar{Y}_y(s,0)\,e^{s\xi} = -\frac{\epsilon\Gamma(0)}{s^2}\frac{\Psi_y(s,0)}{\Psi(s,0)}\,e^{s\xi}. \tag{3.18}$$

It is obvious from physical considerations that the eigenvalues $s_k = \alpha_k + i\beta_k$ of the eigenvalue problem posed above must have negative real parts. Thus the residue of the function given in (3.18) for the pole at the origin $s = 0$ must be such as to account for the pressure rise that is to be achieved by the deflection

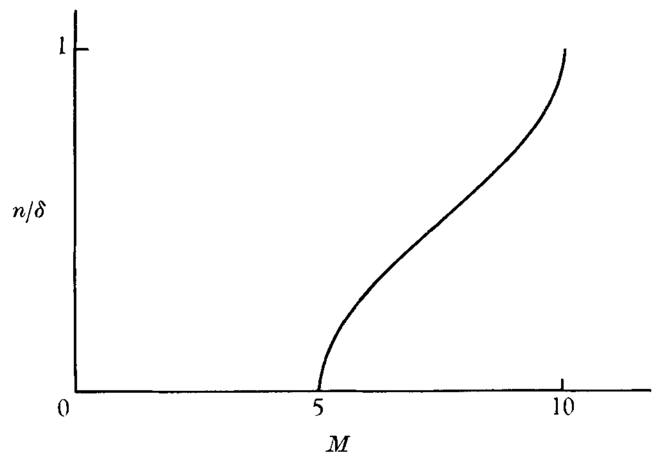

FIGURE 3. Mach-number distribution (equation (3.20)) used in illustrative numerical solution.

through the angle ϵ of the flow outside the entropy layer. The residue of the function $\Psi_y(s,0)\,e^{s\xi}/s^2\Psi(s,0)$ at $s = 0$ is thus $-\Gamma_\infty/\Gamma(0)$. The pressure distribution is then given (if all the poles are simple) by

$$P(\xi,0) = \epsilon\Gamma_\infty - \epsilon\Gamma(0)\sum_k \frac{\Psi_y(s_k,0)}{s_k^2\,\Psi'(s_k,0)}\,e^{s_k\xi}, \tag{3.19}$$

where the s_k are the eigenvalues of the system given by (3.13), (3.14), and (3.17), and $\Psi'(s,0)$ is the derivative with respect to s of the function $\Psi(s,0)$.

As an example of this method of solving for pressure distributions, consider that the entropy distribution above the plate is given by the Mach-number distribution

$$M = 5 + 5(n/\delta)^2\,(3 - 2n/\delta), \tag{3.20}$$

which is illustrated in figure 3. A numerical procedure was used to obtain the eigenvalues and eigensolutions for this case. The positions of the first six poles of the pressure distribution function (3.18) are shown in figure 4. The behaviour of the eigensolutions $\Psi = u + iv$ is shown in figure 5. The pressure distributions obtained by summing the residues at the first, the first two, the first four, and the first six poles of the pressure distribution function are shown in figure 6 where

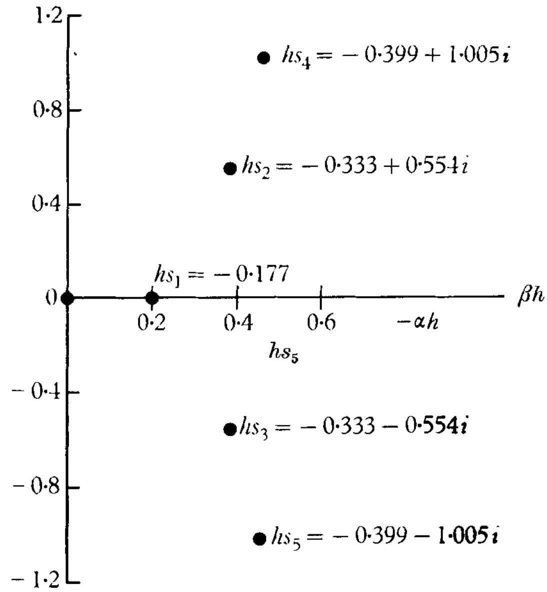

FIGURE 4. Location of the poles in the complex s-plane of the transform of the pressure distribution function for the Mach-number distribution given by equation (3.20).

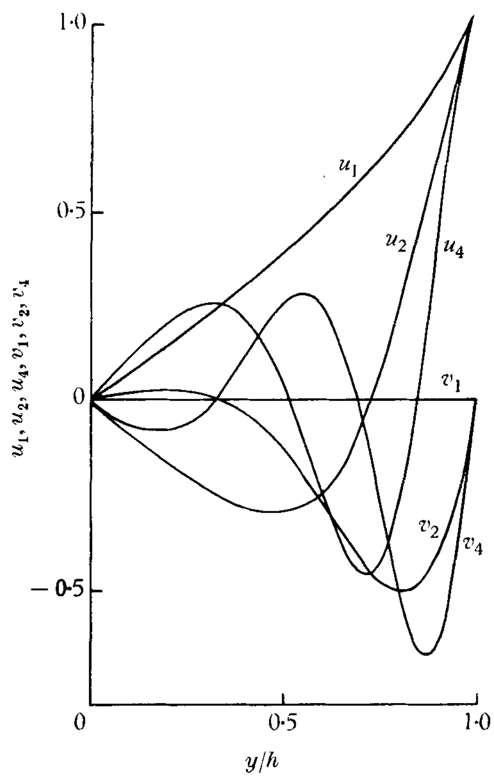

FIGURE 5. Behaviour of eigensolutions $\Psi = u + iv$ for the Mach-number distribution given by equation (3.20).

they are compared with the exact pressure distribution calculated by the method of characteristics.

The character of the pressure distributions obtained by the procedure just outlined may be seen from these results. For Γ-distributions of the type under discussion here, i.e. Γ a non-decreasing function of y, it appears that, in general, the poles consist of: (1) the pole at the origin which accounts for the pressure change far from the corner due to the flow outside the entropy layer; (2) a small negative real pole which accounts, in general, for the scale of distance required for the expansion or compression caused by the corner to be complete; and (3) an infinite set of complex poles having more and more negative real parts which account for more and more details of the pressure distribution, especially near the corner.

From the results just presented, it seems apparent that the character of the flow downstream of a corner for the case of monotone-increasing Γ can, in general, be obtained from the first two poles of the pressure distribution function. Since the pole at the origin and its residue are known, this requires that only one real eigenvalue of the problem posed by equations (3.13), (3.14), and (3.17) need be found. This proves to be a not too difficult problem numerically.* However, if more details of the flow are required, the computational problem becomes formidable as the search for the complex eigenvalues is now two-dimensional, and the character of the eigenfunctions themselves for the higher eigenvalues requires more careful integration techniques as may be seen from figure 5. Furthermore, the convergence is slow. From this it may be concluded that the present method has advantages over the method of characteristic from a computational point of view only when the general character of the flow is required and local details in the vicinity of the corner are not of primary importance.

These remarks may be brought to better focus by considering the behaviour of the pressure distribution due to the step function in Γ considered previously, by summing the residue of a finite number of poles of the pressure distribution function derived from equation (3.7), namely

$$\bar{P}(s,y) = \frac{\epsilon\Gamma_b}{s}\left(\frac{e^{s(y+\xi)}+e^{-s(y-\xi)}}{1-\{(\Gamma_\infty-\Gamma_b)/(\Gamma_\infty+\Gamma_b)\}e^{-2hs}} - e^{s(y+\xi)}\right).$$

The poles of this function are shown in figure 7. The pressure distributions obtained by summing the residue of the pole at the origin, the two real poles, the first four poles about the origin, and the first ten poles about the origin are compared in figure 8, with the exact solution already given. These results make quite clear the nature of these solutions for Γ-distributions which are increasing

* Lighthill (1953) obtains a first approximation to the value of this eigenvalue by assuming it is so small that the second term of (3.13) can be neglected. Then

$$\Gamma\Psi_y = \text{const.} = \Gamma(h)\,\Psi_y(h)$$

or, using (3.14)

$$s_1\Gamma(h)/\Gamma = -\Psi_y/\Psi(h).$$

Integrating this equation, and using (3.17) gives

$$s_1^{-1} = -\int_0^h \frac{\Gamma(h)}{\Gamma}\,dy,$$

which is Lighthill's expression.

functions of y in the region $0 < y < h$. We again see that the general character of the pressure distribution is given by the residues of the two poles closest to the origin and that more and more detail is given by adding the residues of more and more poles. Again it should be pointed out that the method cannot give accurate results in the neighbourhood of the corner unless many terms are taken in the eigenfunction expansion of the solution.

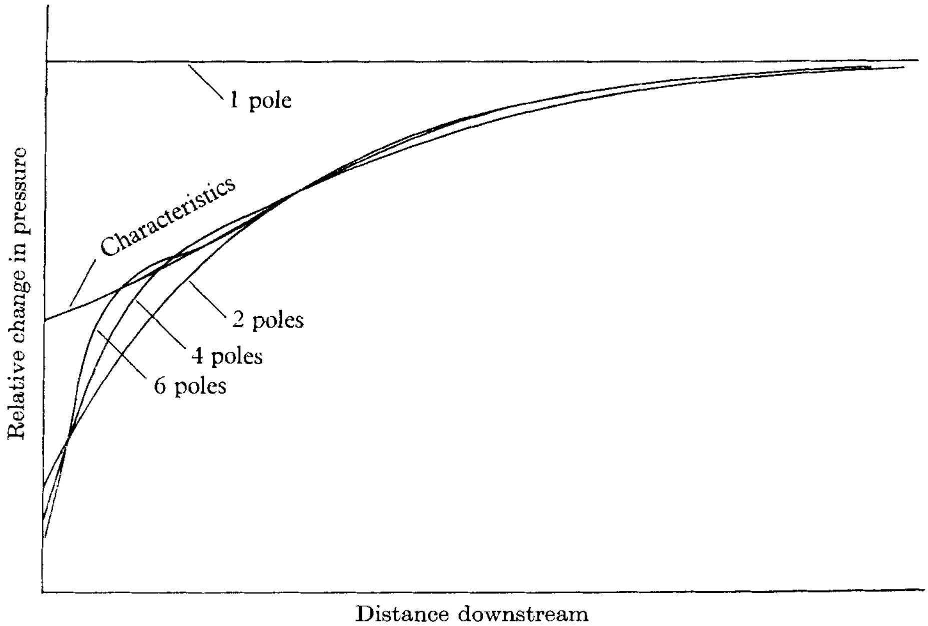

FIGURE 6. Surface pressure distribution downstream from a corner for the Mach-number distribution given by equation (3.20).

The results just presented hold for distributions $\Gamma(y)$ that are monotone increasing functions of y. Somewhat similar results hold for the case of distributions $\Gamma(y)$ that are monotone decreasing functions of y, although the distribution of poles of the pressure distribution function is quite different. In the case $\Gamma' = d\Gamma/dy < 0$, it is not hard to show that except for the pole at the origin there are no poles on the real axis. We multiply equation (3.13) through by Ψ_y/Γ and integrate with respect to y from $y = 0$ to $y = h$ and obtain

$$[\Psi_y(s,h)]^2 - [\Psi_y(s,0)]^2 + 2\int_0^h \frac{\Gamma'}{\Gamma}\Psi_y^2\,dy = s^2[\Psi(s,h)]^2.$$

Using the boundary condition

$$\Psi_y(s,h) + s\Psi(s,h) = 0,$$

we obtain

$$[\Psi_y(s,0)]^2 = 2\int_0^h \frac{\Gamma'}{\Gamma}\Psi_y^2\,dy.$$

Since we have assumed $\Gamma' = d\Gamma/dy < 0$ and $\Gamma > 0$, we see from this expression that Ψ may not be real. This in turn requires that s_k be complex. Nevertheless, it is still true that the poles nearest the origin define the large-scale or gross features of the pressure distribution while the poles far from the origin determine the details or high-frequency components of the distribution as in the case of monotone increasing functions of Γ. One again finds, because of this situation,

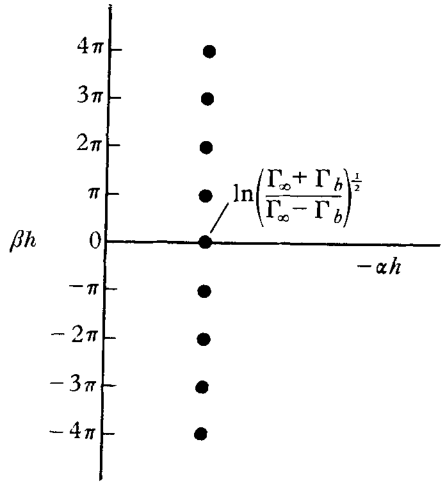

FIGURE 7. Location of the poles is the complex s-plane of the transform of the pressure distribution function for a step Γ-distribution (equation (3.6)) with $\Gamma_\infty > \Gamma_b$.

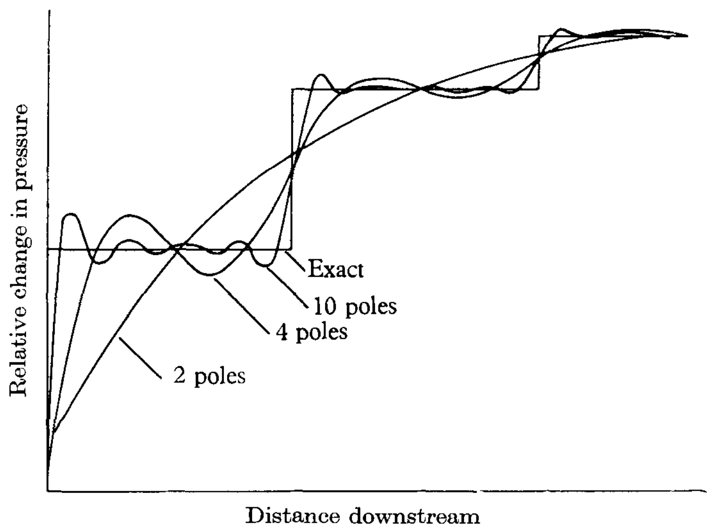

FIGURE 8. Surface pressure distribution downstream from a corner for a step Γ-distribution (equation (3.6)) with $\Gamma_\infty/\Gamma_b = 2$.

that the pressure in the immediate vicinity of the origin is poorly defined. For example, we may observe the behaviour of the pressure distributions obtained by considering the first three, the first five, and the first eleven poles in the pressure distribution function for the case of a step-function Γ-distribution when $\Gamma_\infty < \Gamma(0)$ (see figure 9).

3.4. *Solution near* $\xi = 0$

We have just seen that the method outlined above does not, except by the expenditure of very large effort, yield valid answers in the vicinity of the corner, i.e. for $\xi \to 0$. In many cases, however, it is not the character of the pressure distribution far from the corner that is of interest. For example, in the case of a flap at the trailing edge of a hypersonic airfoil whose chord is not large compared to the thickness of the entropy layer into which it deflects, we do not care about the pressure distribution far from the wing-flap juncture. In this case, we are interested in the pressure distribution near this juncture. We therefore seek a solution for the pressure distribution valid in the vicinity of $\xi = 0$.

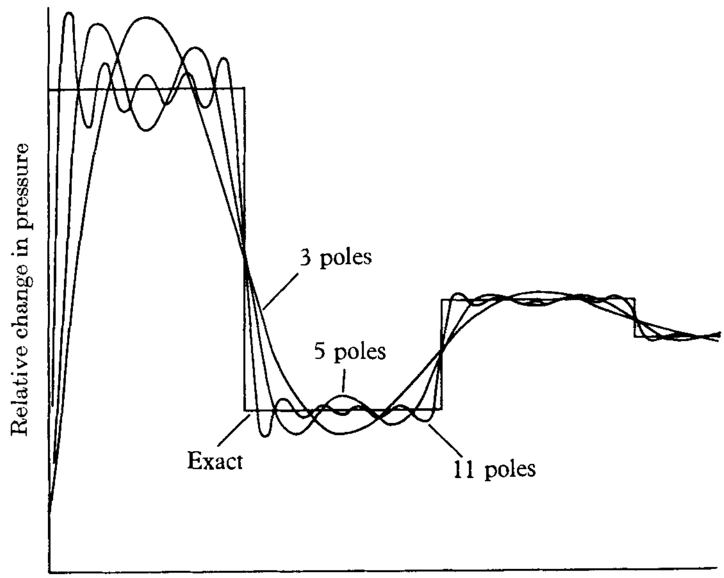

FIGURE 9. Surface pressure distribution downstream from a corner for a backward step Γ-distribution (equation (3.6)) with $\Gamma_\infty/\Gamma_b = \frac{1}{2}$.

If Γ is an analytic function of y, P is an analytic function; if Γ is analytic in the domain $0 \leqslant y < h$, $P(\xi, 0)$ is analytic in the domain $0 < \xi < 2h$ since the first wave that reflects back from $y = h$ arrives at the wall at $\xi = 2h$. Thus, for small ξ the pressure distribution can be expressed as a power series. This procedure is particularly attractive when, for example, the chord of the flap in the example cited above is of the order of h.

In order to develop the series solution new variables are introduced by

$$\sigma = \xi - y, \quad \tau = y, \tag{3.21}$$

$$Z(\sigma,\tau) = Y(\xi,y), \tag{3.22}$$

$$g(\tau) = (\Gamma(y)/\Gamma_b)^{\frac{1}{2}}, \tag{3.23}$$

where $\Gamma_b = \Gamma(0)$. Then equation (2.16) becomes

$$gZ_{\sigma\tau} + g'Z_\sigma = \tfrac{1}{2}gZ_{\tau\tau} + g'Z_\sigma, \tag{3.24}$$

and the third of conditions (2.17) can be written

$$Z(\sigma, 0) = -\epsilon 1(\sigma)\sigma \tag{3.25}$$

for $F(\xi) = \epsilon 1(\xi)$.

An expansion of Z in powers of σ is assumed

$$Z = -\epsilon 1(\sigma)\left[Z_1(\tau)\sigma + Z_2(\tau)\frac{\sigma^2}{2!} + Z_3(\tau)\frac{\sigma^3}{3!} + \dots\right]. \tag{3.26}$$

Substituting this expression into equation (3.24) and equating the coefficients of equal powers of σ, we get

$$\left.\begin{aligned} &gZ_1' + g'Z_1 = 0, \\ &gZ_n' + g'Z_n = \tfrac{1}{2}gZ_{n-1}'' + g'Z_{n-1}' \quad (n > 1); \end{aligned}\right\} \tag{3.27}$$

equation (3.25) becomes

$$\left.\begin{aligned} &Z_1(0) = 1, \\ &Z_n(0) = 0 \quad (n > 1); \end{aligned}\right\} \tag{3.28}$$

Z_1 is immediately found to be $1/g$ since $g(0) = 1$. The other Z_i are not as easily expressed but the $Z_i'(0)$, which are all that is needed to find $P(\xi, 0)$, are readily found when advantage is taken of equations (3.28). Thus it is found that

$$\begin{aligned} Z_1'(0) &= -g'(0), \\ Z_2'(0) &= -\tfrac{1}{2}g''(0), \\ Z_3'(0) &= -\tfrac{1}{4}g'''(0) + \tfrac{1}{4}g'(0)\,g''(0), \quad \text{etc.} \end{aligned}$$

Since

$$P(\xi, 0) = \Gamma_b Y_y(\xi, 0) = \Gamma_b[Z_\tau(\sigma, 0) - Z_\sigma(\sigma, 0)]$$

with $\sigma = \xi$ the series for the wall pressure is

$$P(\xi, 0) = \epsilon\Gamma_b 1(\xi)\left\{1 + g'(0)\,\xi + \tfrac{1}{2}g''(0)\frac{\xi^2}{2!} + [\tfrac{1}{4}g'''(0) - \tfrac{1}{4}g'(0)\,g''(0)]\frac{\xi^3}{3!} + \dots\right\}.$$

The rapidity of convergence of this series, and hence its usefulness for calculations, obviously depends on the particular function $g = (\Gamma/\Gamma_b)^{\frac{1}{2}}$ under consideration.

4. Pressure distribution due to wave impingement

We now consider the second fundmental problem, namely the pressure distribution that results from the impingement of a compression or expansion wave on a flat surface above which exists an entropy layer. The basic equations in this case are

$$(\Gamma\overline{Y}_y)_y - s^2\Gamma\overline{Y} = 0, \tag{4.1}$$

$$\overline{Y}(s, 0) = 0, \tag{4.2}$$

$$\overline{Y}(s, h) + s\overline{Y}(s, h) = -2\overline{G}(s). \tag{4.3}$$

As before we consider the eigenvalue problem defined by

$$(\Gamma\Psi_y)_y - s^2\Gamma\Psi = 0, \tag{4.4}$$

$$\Psi(s, 0) = 0, \tag{4.5}$$

$$\Psi(s, h) + s\Psi(s, h) = 0. \tag{4.6}$$

We note that we may write

$$\bar{Y}(s,y) = \phi(s)\,\Psi(s,y) \tag{4.7}$$

and

$$\bar{Y}_y(s,y) + s\,Y(s,y) = \phi(s)\,[\Psi_y(s,y) + s\Psi(s,y)]. \tag{4.8}$$

From equation (4.8) we note that

$$\phi(s) = \frac{-2\bar{G}(s)}{\Psi_y(s,h) + s\Psi(s,h)}, \tag{4.9}$$

and hence from equation (4.7)

$$\bar{Y}(s,y) = \frac{-2\bar{G}(s)\,\Psi(s,y)}{\Psi_y(s,h) + s\Psi(s,h)}. \tag{4.10}$$

The transform of the surface pressure distribution is then

$$\bar{P}(s,0) = -\frac{2\Gamma(0)\,\bar{G}(s)\,\Psi_y(s,0)}{\Psi_y(s,h) + s\Psi(s,h)}. \tag{4.11}$$

It is seen from this expression that the poles of this function are determined by precisely the same eigenvalue problem as in the case of determining the pressure distribution due to a deflection of the surface. The eigenvalues (poles) are therefore identical. Indeed, the whole problem is so similar to those just discussed that a detailed discussion need not be undertaken; two simple examples will suffice.

For the two special Γ-distributions considered in § 3.2 and for a simple incoming wave of the type

$$g(\xi+y) = \epsilon 1(\xi+y-h)$$

the wall pressure is easily found. Thus

$$P(\xi,0) = -2\epsilon\Gamma_\infty(1-G^{n+1}),$$

when $(2n+1)h < \xi < (2n+3)h$ for

$$\Gamma = \Gamma_b \quad (0 \leqslant y < h),$$
$$\Gamma = \Gamma_\infty \quad (h < y),$$

and

$$P(\xi,0) = -2\epsilon\Gamma_b\frac{a+h}{a}\sum_{k=0}^{n}(-1)^k\frac{X_k^k}{k!}e^{X_k}$$

when $(2n+1)h < \xi < (2n+3)h$ for

$$\Gamma = \Gamma_b(1+y/a)^2 \quad (0 \leqslant y \leqslant h),$$
$$\Gamma = \Gamma_b(1+h/a)^2 = \Gamma_\infty \quad (h \leqslant y).$$

5. Summary

In the preceding sections we have presented a discussion of the pressure distributions which result when small perturbations are made in an otherwise undisturbed, planar, inviscid, supersonic flow in which a strong entropy layer exists. The complete solution of this problem can be obtained explicitly only for a few very special distributions of entropy. In general, however, the character of such pressure distributions can be obtained in the form of an eigenfunction expansion of the solution. The general method is useful for it allows the user to determine in a fairly simple fashion the behaviour and scale of the interactions produced by the penetration of an entropy layer by a pressure wave. For the fine structure of the pressure distributions that result from such interactions, the computational effort required to obtain valid results is such that one would, in general, choose to use the more general and exact method of characteristics for rotational flow. For certain special cases, where one is interested in the detailed behaviour of the interaction only near the start of such an interaction, these details can be found in the form of a series solution for the pressure distribution that is valid throughout the region of interest. Probably the most useful aspect of the present theory is not to be found in its use as a computational tool, but rather in the physical insight it affords into a very basic phenomena of hypersonic flow, namely, the interaction of pressure waves with flow layers which have strong variations of entropy.

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